A_4 Has No Subgroup of Order 6

Proposition. *The alternating group A_4 has no subgroup of order 6.*

Proof. Let $G \cong A_4$, let $H \leq G$, and to obtain a contradiction, suppose $|H| = 6$. We begin by establishing that

$$a^2 \in H \quad \text{for all} \quad a \in G. \tag{1}$$

If $a \in H$, then this is clear (since H is closed under the group operation). Assume, therefore, that $a \notin H$. Then aH is a left coset distinct from $1H = H$, and since $|G : H| = 2$, these are the only two left cosets of H in G:

$$G = aH \cup H \quad \text{and} \quad aH \cap H = \varnothing.$$

Suppose $a^2 \in aH$. Then $a^2 = ah$ for some $h \in H$, and it follows that $a = h \in H$ (after multiplying on the left by a^{-1}), a contradiction. This proves (1).

To complete the proof, observe that if $\sigma = (a\ b\ c)$ is any 3-cycle in G, then $\sigma = (\sigma^2)^2$ (where $\sigma^2 = \sigma^{-1}$ is also a 3-cycle). Hence every 3-cycle is the square of an element of G, and it follows by the argument above that every 3-cycle in G is contained in H. But there are eight 3-cycles in A_4, and H has only six elements. This contradiction proves that no such subgroup H exists. □