The
Classification
of the
Finite Simple Groups

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What is it?

A theorem:

- Proven in 10 to 15 *thousand* journal pages; about 500 articles.
- The work of over 100 mathematicians, mostly between 1950 and 1980.
- Completed in 1983 (or 2004. . .).

The Celebration

Actually. . .

Groups

A group is a set G with a binary operation such that:

- There is an *identity* element,
- Every element has an *inverse*,
- The operation is *associative*.

Example: The integers $\mathbb{Z}$ under addition.

Subgroups

A subgroup is a subset of G that is itself a group under the same operation.

Example: $3\mathbb{Z} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$

Notation: $3\mathbb{Z} \leq \mathbb{Z}$

Quotient Groups

Example: Partition $\mathbb{Z}$ into *equivalence classes* by identifying integers that differ by an element of $\mathbf{3}\mathbb{Z}$:

$$\begin{aligned}
\overline{\mathbf{0}} &= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} &&= 3\mathbb{Z} \\
\overline{\mathbf{1}} &= \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\} &&= 1 + 3\mathbb{Z} \\
\overline{\mathbf{2}} &= \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\} &&= 2 + 3\mathbb{Z}
\end{aligned}$$

These classes form a group, $\mathbb{Z}/\mathbf{3}\mathbb{Z}$, under addition.

Isomorphism

Let Z_3 be the multiplicative group $\{1, x, x^2\}$ where $x^3 = 1$.

Z_3

$\cdot$	1	x	x^2
1	1	x	x^2
x	x	x^2	1
x^2	x^2	1	x

Isomorphism

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$\mathbb{Z}/3\mathbb{Z}$

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

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Z_3

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$\mathbb{Z}/3\mathbb{Z}$ is *isomorphic* to Z_3

Simple Groups

If the quotient G/N is a group, then N is a *normal* subgroup.

A group with no normal subgroups is *simple*.

Example: Z_3 is simple.

The Idea

Theorem (Jordan – Hölder). *Every finite group can be "factored" into simple groups in an essentially unique way.*

Thus the finite simple groups are the "periodic table" of atomic elements of finite groups.

Hölder Program (1892):

- Classify the finite simple groups.
- Find all the ways to combine them to form other groups.

First Big Breakthrough

Notation and Terminology:

- $|G|$: The number of elements in G.
- t is an *involution* if t^2 is the identity.
- $C_G(t)$: the subset of G that commutes with t.

Theorem (Brauer – Fowler, 1955). $|G| \leq (2|C_G(t)|^2)!$

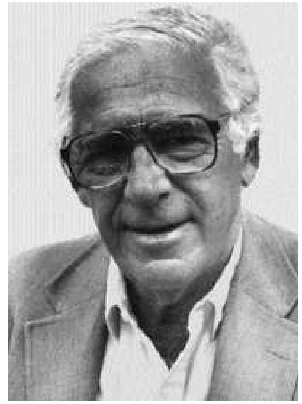
Implication: There are only finitely many groups with a given involution centralizer.

Second Big Breakthrough

Theorem (Feit – Thompson, 1963). *Odd order groups are solvable.*

Implication: Except for the groups Z_p, every finite simple group has even order.

Lots of Hard Work



1972: Daniel Gorenstein begins handing out tasks. . .

1983: Daniel Gorenstein announces the classification.

2004: Aschbacher and Smith complete the last case.

The Classification

Theorem. *Every finite simple group is*

- *cyclic of prime order,*
- *an alternating group,*
- *a finite simple group of Lie type, or*
- *one of twenty-six sporadic finite simple groups.*

The Groups of Lie Type

(i) the linear groups $A_n(q) = PSL_{n+1}(q)$

(ii) the unitary groups ${}^2A_n(q) = PSU_{n+1}(q)$

(iii) the orthogonal groups of odd dimension $B_n(q) = P\Omega_{2n+1}(q)$

(iv) the Suzuki groups ${}^2B_2(q) = Sz(q)$

(v) the symplectic groups $C_n(q) = PSp_{2n}(q)$

(vi) the orthogonal groups of "plus" type in even dimension $D_n(q) = P\Omega^+_{2n}(q)$

(vii) the orthogonal groups of "minus" type in even dimension ${}^2D_n(q) = P\Omega^-_{2n}(q)$

(viii) ${}^3D_4(q)$

(ix) $G_2(q)$

(x) the Ree groups ${}^2G_2(q) = Re(q)$

(xi) $F_4(q)$

(xii) the Ree groups ${}^2F_4(q) = Re(q)$

(xiii) $E_6(q)$

(xiv) ${}^2E_6(q)$

(xv) $E_7(q)$

(xvi) $E_8(q)$

The Groups of Lie Type

(i) the linear groups $A_n($ …

(ii) the unitary groups ^2A …

(iii) the orthogonal groups … $_{+1}(q)$

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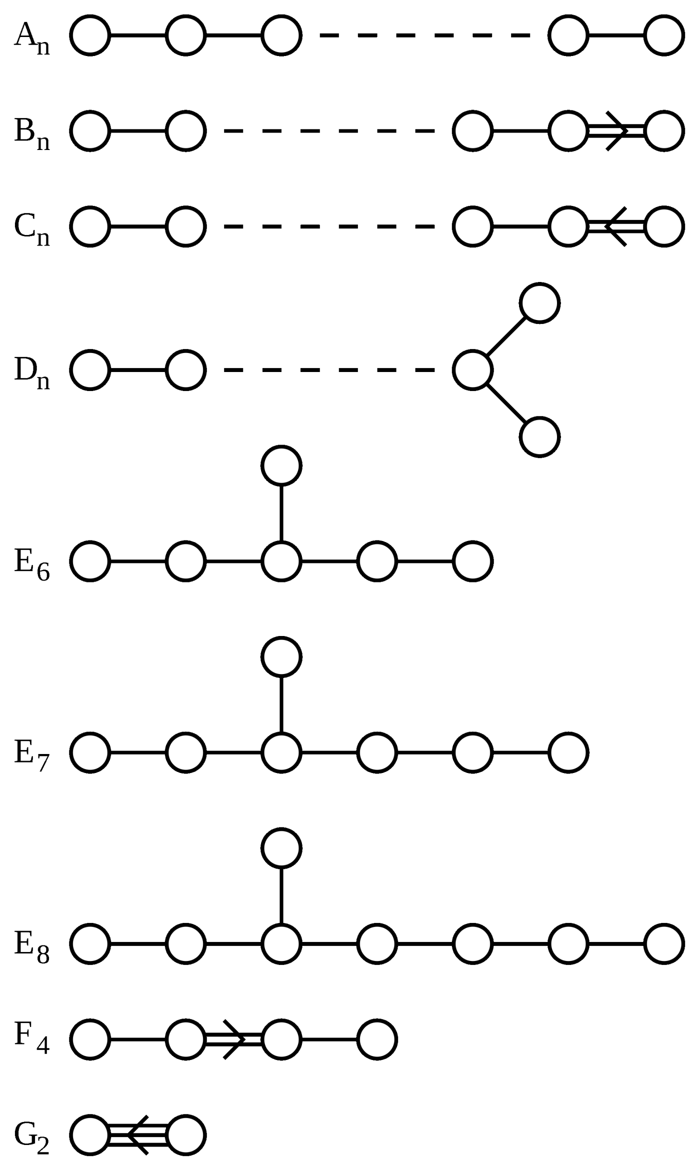

The Sporadics

- the Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24};
- the Janko groups J_1, J_2, J_3, and J_4;
- the Higman-Sims group HS;
- the Held group He;
- the McLaughlin group Mc;
- the Suzuki group Suz;
- the Lyons group Ly;
- the Rudvalis group Ru;
- the O'Nan group $O'N$;
- the Conway groups Co_1, Co_2, and Co_3;
- the Fischer groups Fi_{22}, Fi_{23}, and Fi'_{24};
- the Harada-Norton group HN;
- the Thompson group Th;
- the Baby Monster B;
- the (Fischer-Griess) Monster M.

The Monster

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

$$= 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000$$

$$\approx 8 \cdot 10^{53}$$

Significance

Mathematics: Number theory (Galois groups), Topology (classifying spaces), Geometry (symmetry groups).

Physics: Quantum mechanics, chemistry.

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The Payoff: Part I

Prove theorems by:

1. Reduce to simple groups, and
2. Check.

The Revision

1994: Gorenstein, Lyons, and Solomon begin the Revision of the Classification. To date: 6 of a projected 12 volumes.

The Payoff: Part II

Prove theorems by:

1. Reduce to simple groups, and
2. Check.

Then:

3. Prove *without* invoking the Classification.

For Example...

Theorem. *Let* $P = P_1 \times P_2 \cong Z_3 \times Z_3$ *be a self-centralizing Sylow 3-subgroup of the finite group* G*, and suppose that* P_1 *and* P_2 *are strongly closed. Then*

(1) $\langle {P_1}^G \rangle \cap \langle {P_2}^G \rangle \leq O_{3'}(G)$*;*

(2) $G = G_1 G_2$ *with* $G_i \trianglelefteq G$ *and* $P_i \in \mathrm{Syl}_3(G_i)$ *for* $i = 1, 2$*;*

(3) $G/O_{3'}(G) = \overline{G_1} \times \overline{G_2}$*; and*

(4) $C_{G_i}(P_i) = P_i$ *for* $i = 1, 2$*.*

If moreover $O_3(G) = 1$*, then* $G_i = \langle {P_i}^G \rangle$ *for* $i = 1, 2$*.*

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(4) $C_{G_i}(P_i) = P_i$ *for* $i = 1, 2$.

If moreover $O_3(G) = 1$, *then* $G_i = \langle {P_i}^G \rangle$ *for* $i = 1, 2$.

Given some conditions on the subgroups of order 3,

G splits into two normal subgroups –

In particular, G is not simple.

Thank You

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