The First Isomorphism Theorem

Recall that two groups, G_1 and G_2, are *isomorphic* if there is a bijection f from one to the other that preserves the group operation: $f(xy) = f(x)f(y)$ for all $x, y \in G_1$. This is the same as saying that $G_1 \cong G_2$ if and only if there is a homomorphism from one to the other that is a bijection.

Theorem. *Let $\phi : G \to H$ be a homomorphism. Then $G/\ker\phi \cong \phi(G)$.*

Proof. For notational convenience, let $K = \ker\phi$. Define $\psi : G/K \to \phi(G)$ by $\psi(gK) = \phi(g)$. We will show that ψ is a bijective homomorphism.

Let $a, b \in G$. Then

$$\psi(aK \cdot bK) = \psi((ab)K) = \phi(ab) = \phi(a)\phi(b) = \psi(aK)\psi(bK),$$

so ψ is a homomorphism.

It is clear that ψ is surjective: If $x \in \phi(G)$, then $x = \phi(g)$ for some $g \in G$, and $\psi(gK) = \phi(g) = x$.

To see that ψ is well defined, suppose $aK = bK$. Then $b^{-1}a \in K = \ker\phi$, so $\phi(b^{-1}a) = 1_H$. It follows that $\phi(b^{-1})\phi(a) = (\phi(b))^{-1}\phi(a) = 1_H$, hence $\phi(a) = \phi(b)$, so $\psi(aK) = \psi(bK)$.

The proof that ψ is injective is the same argument in reverse: Suppose $\psi(aK) = \psi(bK)$. Then $\phi(a) = \phi(b)$, so $(\phi(b))^{-1}\phi(a) = 1_H$. It follows that $\phi(b^{-1}a) = 1_H$, so $b^{-1}a \in \ker\phi = K$. Hence $aK = bK$, so ψ is injective. $\square$