## Proof of the First Limit Law

We will need the *Triangle Inequality*: $|a+b| \leq |a| + |b|$

(Easily proven by considering the three cases: $a$ and $b$ are both positive, both negative, or of opposite signs.)

The First Limit Law is: If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Equivalently:

**Proposition:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}[f(x)+g(x)] = L + M$.

*Proof.* We must show that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|(f(x)+g(x)) - (L+M)| < \epsilon \quad \text{whenever} \quad |x-a| < \delta,\ x \neq a.$$

Let $\epsilon > 0$ be given. Then $\frac{\epsilon}{2} > 0$ as well, hence since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ whenever $|x-a| < \delta_1$, $x \neq a$. Similarly, there exists $\delta_2 > 0$ such that $|g(x) - M| < \frac{\epsilon}{2}$ whenever $|x-a| < \delta_2$, $x \neq a$. Choose $\delta = \min(\delta_1, \delta_2)$ (so that $|x-a| < \delta$ implies $|x-a| < \delta_1$ and $|x-a| < \delta_2$). Observe:

$$\begin{aligned} |(f(x)+g(x)) - (L+M)| &= |(f(x)-L) + (g(x)-M)| \\ &\leq |f(x)-L| + |g(x)-M| \end{aligned}$$

by the Triangle Inequality. And if $|x-a| < \delta$ with $x \neq a$, then $|f(x)-L|$ and $|g(x)-M|$ are both less than $\frac{\epsilon}{2}$, so their sum is less than $\epsilon$, as required to complete the proof. □