**Homework 3, due Friday, February 14th:**

**Do, but do not hand in:**

1. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$. Find $|A|$, $|B|$, and $|AB|$.

2. Let $G = \langle a \rangle$ with $|G| = n$. Prove that $g^n = 1$ for every $g \in G$.

   [Hint: Use the lemma from the handout "Orders of Elements of Cyclic Groups": *If* $|x| = n < \infty$, *then* $x^k = 1$ *if and only if* $n \mid k$.]

3. Give the full subgroup lattice for $Z_{12} = \langle a \rangle$.

4. Write each of the following as a product of disjoint cycles:

   (a) $(1\ 2\ 3\ 4)(1\ 3\ 5)$

   (b) $(1\ 2\ 3\ 4)(1\ 2\ 3)$

   (c) $(1\ 5)(1\ 4)(1\ 3)(1\ 2)$

   (d) $(1\ 2\ 3\ 4)(1\ 3)$

   (e) $(1\ 2\ 3\ 4)(1\ 4\ 3)$

   (f) $(1\ 2\ 3\ 4)(3\ 4\ 5\ 6)$

5. Find the order of each of the permutations in exercise 4.

6. Determine whether each of the permutations in exercise 4 is even or odd.

7. Prove that if $|x| = n$, then $x^{n-1} = x^{-1}$.

(Exercises to hand in are on the back...)

**Hand in:**

1. Prove that $|x^{-1}| = |x|$ for any element $x$ of any group $G$.

   [Hint: Assume $|x| = n$ and consider $(x^{-1})^n \, x^n$.]

2. Prove that $S_n$ is non-abelian for all $n \geq 3$, and that $A_n$ is non-abelian for all $n \geq 4$.

3. Prove that $A_8$ contains a cyclic subgroup of order 15.

4. Let $G$ be a group of permutations on a set $\Omega$. Let $a \in \Omega$, and define *the stabilizer in $G$ of $a$* as:
   $$G_a = \{\sigma \in G \ : \ \sigma(a) = a\}.$$
   Prove that $G_a$ is a subgroup of $G$.

   **Note:** Gallian calls this subgroup "stab($a$)." His solution, which is probably in your textbook (and which you are welcome to look at), is both terse and incomplete. Make sure to give a complete solution in your own words.

5. Recall that the *dihedral group* $D_n$ (in Gallian's notation) is defined as the group generated by two elements, $r$ and $s$, such that $|r| = n$, $|s| = 2$, and $rs = sr^{-1}$. In symbols:
   $$D_n = \left\langle r, s \ \mid \ r^n = s^2 = 1, \ rs = sr^{-1} \right\rangle.$$
   (Although we have not proven it formally, our argument in class that $|D_4| = 8$ is easily adapted to show that in general $|D_n| = 2n$.)

   Let $G$ be any group, let $a$ and $b$ be distinct elements of $G$ with $|a| = |b| = 2$, and suppose $|ab| = n$. Prove that $\langle a, b \rangle$, the group generated by $a$ and $b$, is in fact the dihedral group $D_n$.

   [Hints: Observe that $ab \in \langle a, b \rangle$, so $\langle a, b \rangle$ contains an element of order 2 and an element of order $n$. To show that this group is in fact $D_n$ requires establishing that the relation "$rs = sr^{-1}$" holds.]

*Please typeset in LaTeX your solution to exercise 4.* You may use http://www.hyginsberg.com/docs/hw3_template.tex as a template, if you like.