Homework 4, due Friday, February 28th:

Do, but do not hand in:

1. Find all of the left cosets of the subgroup $H = \langle(1\ 2\ 3)\rangle$ of A_4.
2. How many left cosets of the subgroup $H = \langle(1\ 2\ 3\ 4)\rangle$ are there in S_6?
3. Suppose a group G contains elements of every order from 1 through 10. What is the minimum possible order of G?
4. Let p be a prime and let G be a group of order p^2. Prove that either G is cyclic or $g^p = 1$ for every $g \in G$.
5. Suppose H and K are subgroups of G with $|H| = 12$ and $|K| = 35$. Find the order of $H \cap K$.

(Exercises to hand in are on the back...)

Notation: For any subset $S = \{x_1, x_2, \ldots, x_n\}$ of any group G, we call the smallest subgroup containing S the *subgroup generated by* S, denoted $\langle x_1, x_2, \ldots, x_n \rangle$.

(This is an extension of the familiar notation for cyclic groups and subgroups, which are generated by a single element; you have also seen this in the definition of D_n which is $\langle r, s \rangle$ subject to some additional conditions.)

Hand in:

1. Prove that if H and K are both normal subgroups of G, then $H \cap K$ is a normal subgroup of G.

2. Let H and K be subgroups of G, and define
$$HK = \{hk \mid h \in H \text{ and } k \in K\}.$$
   (a) Let $G \cong S_3$, $H = \langle (1\ 2) \rangle$, and $K = \langle (2\ 3) \rangle$. Find HK (i.e. list all of the elements as a set) and explain why HK is *not* a subgroup of G.
   (b) Prove that if $H \trianglelefteq G$, then HK is a subgroup of G.
   (c) If we replace the condition "$H \trianglelefteq G$" with "$K \leq N_G(H)$," how must your argument in (b) change? State the theorem that results.

3. Let p and q be distinct primes, and let G be a group of order pq. Prove that if x and y are nonidentity elements of G and $|x| \neq |y|$, then $\langle x, y \rangle = G$.

4. Let $G \cong S_4$, and consider the permutations $\sigma = (1\ 2)$ and $\tau = (2\ 3\ 4)$ in G. In this exercise, we will prove that $\langle \sigma, \tau \rangle = G$.
   (a) Show (by direct calculation) that $\langle \sigma, \tau \rangle$ contains a 2-2-cycle; call it ω.
   (b) Argue that $\langle \tau, \omega \rangle$ is a subgroup of G isomorphic to A_4.
   (c) Complete the argument that $\langle \sigma, \tau \rangle = G$.

5. Let $H \leq G$. Prove that $C_G(H) \trianglelefteq N_G(H)$.

   (We proved in class that $C_G(H) \leq N_G(H)$, so you need only prove that $C_G(H)$ is *normal* in $N_G(H)$).

Please typeset in LaTeX your solution to exercise 5.