Homework 6, due Friday, March 27th:

Do, but do not hand in:

1. Find a counterexample in a ring $\mathbb{Z}/n\mathbb{Z}$ to each of the following common principles of the algebra of real numbers:

   (a) If $ab = 0$, then $a = 0$ or $b = 0$.

   (b) If $a^2 = a$, then $a = 0$ or $a = 1$.

   (c) If $ab = ac$, then $b = c$.

2. Prove that $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \ : \ a, b \in \mathbb{Z}\}$ is an integral domain.

   (Hint: To prove that it's a ring, observe that $\mathbb{Z}[\sqrt{5}] \subseteq \mathbb{R}$ and use the subring test.)

(Exercises to hand in are on the back...)

Hand in:

1. Let a be an element of a ring R. Define the *right annihilator* of a as the set

$$\mathrm{Ann}(a) = \{x \in R \ : \ ax = 0\}.$$

   Prove that $\mathrm{Ann}(a)$ is a subring of R.

2. Let R be a ring and let $a \in R$. Prove that if a is a unit, then a is not a zero divisor.

3. For any rings R and S, the set

$$R \oplus S = \{(r, s) \ | \ r \in R \text{ and } s \in S\}$$

   is a ring under the operations of *componentwise* addition and multiplication, i.e.:

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \quad \text{and} \quad (r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2)$$

   for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$.

   Describe the set Z of all zero divisors and the set U of all units in the ring $\mathbb{Z} \oplus \mathbb{Q}$.

4. Prove that the ring $(\mathbb{Z}/5\mathbb{Z})[i] = \{a + bi \ : \ a, b \in \mathbb{Z}/5\mathbb{Z}\}$ (where i is the usual imaginary unit, $i = \sqrt{-1}$) is *not* an integral domain.

5. Prove that $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \ : \ a, b \in \mathbb{Q}\}$ is a field.

Please typeset in LaTeX your solution to exercises 1 and 2 (at least).