How to Write Proofs in MA-405

(Version 2; February 2nd, 2020)

Introduction: It goes without saying that of course your proofs should be logically correct, and I have said explicitly in the syllabus that I want your proofs presented in well-written sentences, organized into paragraphs, and that we will always be striving to write *beautiful* mathematics. But there are nuances to proof writing that change as you progress from the most basic courses (such as *Theory of Proof*), to intermediate courses (such as *Number Theory* and *Modern Geometry*) to more advanced courses such as this one. This document is my attempt to state as explicitly as I can what is allowed and even expected at your current presumed level of what is known as "mathematical maturity."

I will begin to list details in a moment, but the guiding principle is *concision.* Say what needs to be said to present a solid argument, say it well, and *please* leave out everything else. When I am reading your proof I will undoubtedly be reading a *stack* of similar efforts, and my patience for reading unnecessary explanations and statements of the obvious wears out quickly…

One last formality: By virtue of your satisfying the prerequisites to enroll in this 400-level class, and by my authority as a recipient of the Doctor of Philosophy (Ph.D.) degree in Mathematics, I hereby confer upon you the mathematical maturity level of "undergraduate advanced." (The continued advancement to ever higher levels of "mathematical maturity" is the mathematician's lifelong pursuit.) This excuses you from some of the requirements that were imposed upon you by teachers in lower level classes, and gives you the freedom – and the responsibility – to begin writing in a manner approaching that of a professional mathematician.

Okay – the longest ever introduction to a document that is purportedly about "concision..." We begin:

1. If you are writing a direct proof of a conditional statement of the form "If P then Q," then *of course* you are assuming that P is true (and showing that Q follows). Please do not restate the hypotheses! (If you are using the contrapositive, or proof by contradiction, or some other technique, then it will be necessary to explicitly state what it is that you are assuming, but if you are assuming precisely what the proposition says to assume, then just take that as understood and *begin.*)

   As a caveat here, if you are proving a *biconditional* statement – an "if and only if" – then there are *two* conditional statements to prove, and you *absolutely must* state your assumptions to let the reader know which of the two conditional statements you are addressing. (When you switch to the other, it is good form to start a new paragraph and alert the reader with a phrase such as "For the converse…" State your assumptions for the converse as well, to make it clear that the assumptions from the previous argument no longer apply.)

2. Real mathematicians do not mention the contrapositive, we simply *use it*. Here again it comes down to the statement of assumptions: when you explicitly assume that the conclusion of a conditional statement is false (i.e. assuming "not Q" for a statement of the form "If P then Q"), you alert the reader that you are using the contrapositive, and that is all the indication you should give. Reach the conclusion "not P" and then declare victory (and *still* do not mention the contrapositive).

3. Real mathematicians are divided on whether or not they mention contradiction at the start of a proof using that technique. It is enough to explicitly assume the negation of the proposition, reach a contradiction, *say so* (at some point the word "contradiction" must appear), and conclude that the proof is therefore complete. Personally I find it kinder to the reader to give some advance warning, and am partial to starting such proofs with "Assume, to obtain a contradiction, that...," but it is your call. You *must*, however, have a clear understanding that this is the technique you are using – a rambling argument that happens to end with a contradiction is more likely to simply be incorrect than it is to be a valid proof by contradiction.

   (Often a small result in the middle of a complex argument warrants a proof by contradiction, and in such cases it is more common to see a less formal treatment, where the assumptions are not explicitly described as being made "to obtain a contradiction.")

4. Very occasionally, in the context of a complex argument, it might be appropriate to explain to the reader what it is that must be shown to complete the proof or some part of the proof. But if your "we must show" statement is really just a note to yourself, a reminder of where you're trying to go with your argument, then please leave it out – it belongs on your scrap paper, not in your proof.

5. I cannot think of any case where the phrase "by the definition of" is appropriate. You have to know the pertinent definitions, and you have to assume your reader does too; just *use* the definitions, don't state them or even refer to them. For example, "Since x is even, by the definition of even numbers $x = 2k$ for some integer k" should just be "Since x is even, $x = 2k$ for some integer k."

6. Leave out any statements that, while true, do not advance your argument. Yes, when you are working out your proof on your scrap paper, you might write down all kinds of things that are true and even interesting, but once you find a sequence of those statements that establishes the truth of your proposition, get rid of any statements that did not actually contribute to your argument.

7. As an “undergraduate advanced” mathematician you have accrued a certain amount of mathematical clout, and you no longer need to show and justify every step of every algebraic computation. You can write $(a + b)^2 = a^2 + 2ab + b^2$ without explaining

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

(Do you see how *painful* and *tedious* that was to follow? – It’s all true, none of it is wrong, it’s even good, and we’re *past* that now. . .) It’s even worse if you want to tell me why each step is true – citing the distributive and associative and commutative properties. . . Please don’t.

I will end by noting that there will be judgment calls to be made here. When we begin ring theory and you are trying to prove from basic principles that $x \cdot 0 = 0$ for all x, for example, it might make perfect sense – and even be important – to cite just the kinds of steps and arguments that I told you to omit in the last item above. That’s okay; you will figure it out. . .

In the worst case, if your judgment differs significantly from mine, I won’t pretend this document will give you any recourse – you will pay a price. . . Try not to worry about it; I have a generous rewrite policy, and you can regain most of any points you lose. And maybe your proof-writing will improve a little as a result.

A final word of warning and apology: Proof writing, like all writing, is both personal and subjective, and for every “rule” I propose there will be (very rare) occasions when it should be ignored, as well as respected mathematicians who will assert the exact opposite. Feel free to complain about it amongst yourselves or to me; I offer in my defense that this document contains the best advice I could muster.

And don’t forget to have fun.

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